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Nonlinear Oscillations

1D

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I.) Nonlinear Oscillations – An Introduction

a.) Overview / Roadmap

Can formulate sequence of problems:

1.) Simple harmonic oscillator – linear

→ external force < deterministic, random

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x = f_{ext}$$

α: damping (friction const.)

ω_0^2 → k/m

if $f_{ext} = f e^{i\gamma t} \Rightarrow \begin{cases} x = A\, e^{i\gamma t} \\ A = f/m(\omega_0^2 - \gamma^2 + 2i\alpha\gamma) \end{cases}$

2.) Variation: SHO (Linear), with Parametric Variation

i.e. $\omega_0^2 = \omega_0^2(t)$

$\omega_0^2(t)$ <

- slow variation $\frac{1}{\omega_0^2}\frac{d}{dt}\omega_0^2 \ll \omega_0 \Rightarrow$ adiabatic theory
- fast variation $\frac{1}{\omega_0^2}\frac{d}{dt}\omega_0^2 > \omega_0$, $\sim 2\omega_0 \Rightarrow$ parametric instability

For parametric instability, prototype system is:

$$\begin{cases} \ddot{x} + \omega^2(t)\,x = 0 \\ \omega^2(t) = \omega_0^2(1 + h\cos\gamma t) \\ \gamma \sim 2\omega_0 + \epsilon \quad , \quad \epsilon \ll \omega_0 \end{cases}$$

$\begin{cases} \omega^2 = \omega^2(t) \\ \Rightarrow \partial L/\partial t \neq 0 \\ \Rightarrow \text{energy not const., here} \end{cases}$

Mathieu's Eqn.

$\begin{cases} \text{instability for} \\ \gamma \cong 2\omega_0 \text{ resonance} \end{cases}$

then true nonlinear problems.

3.) Nonlinear Oscillator - Conservative

i.e. $\ddot{x} + \omega_0^2 x + \underbrace{\epsilon x^3}_{\text{anharmonic / nonlinear term}} = 0$ Duffing's Equation

Here, ⇒ conservative system, with bounded phase space trajectories

i.e. $\frac{1}{2}\dot{x}^2 + F(x) = E$ → quasi-periodic orbits bounded

$F(x) \to \omega_0^2 x^2/2 + \epsilon x^4/4$

→ secularity in perturbation theory

$$\ddot{x} + \omega_0^2 x + \epsilon x^3 = 0$$

$$\overline{\omega}^2 = \frac{\omega_0^2 \langle x^2\rangle + \epsilon\langle x^4\rangle}{\langle x^2\rangle} \quad \leftarrow x^{(0)} \sim a \sin\omega_0 t$$

$\epsilon x^3 \sim a^3 \sin^2\omega_0 t \sin\omega_0 t$

$\sim a^3 \sin\omega_0 t$ (on resonance)

⇒ secularity $\sim t\cos\omega_0 t$?!?

∴ ⇒ nonlinear frequency shift ! ⟷ removes secularity!

→ simplest example of Reductive P.T.

$$\Rightarrow \quad \omega = \omega_0^0 + \omega^{(2)}$$

i.e. $\omega_0^2 x + \epsilon x^3$

$\rightarrow (\omega_0^2 + \epsilon \overline{x^2})x$

$\sim (\omega_0^2 + (\#)\epsilon a^2)\, x$

$\mapsto$ effective ω-correction!

2nd-order correction $\sim \dfrac{3\epsilon}{8\omega_0} a^2$ amplitude dep.

4.) Forced Nonlinear Oscillator

i.e. $\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \epsilon x^3 = f_{ext}(t)$ — Forced Duffing's Equation

ϵx^3: NL; $f_{ext}(t)$: forcing

Here $\rightarrow$ role of nonlinearity / anharmonicity in resonance

$\rightarrow$ i.e. contrast: forced SHO $(\epsilon = 0)$

$f_{ext} = f e^{i\omega t}$; $x = B e^{i\omega t}$

$B = b e^{i\delta}$

dissipation resolves resonance

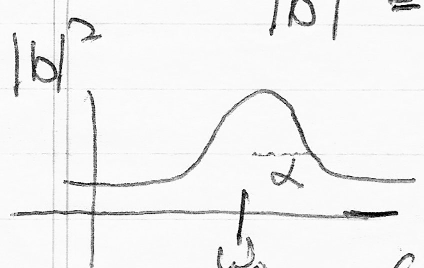
$$b = f/m \left[(\omega_0^2 - \omega^2)^2 + 4\alpha^2\omega^2\right]^{1/2}$$

$$\tan\delta = 2\alpha\omega / (\omega^2 - \omega_0^2)$$

so:

$$|b|^2 = f^2/m^2\left[(\omega_0^2-\omega^2)^2 + 4\alpha^2\omega^2\right] \rightarrow f^2/4m^2\omega_0^2 \; {}^{*}\left(1/(\omega_0-\omega)^2+\alpha^2\right)$$

(near resonance)



resonance $\rightarrow \omega_0$

linewidth $\rightarrow \alpha$

(symmetric)

but for NL forced SHO;

→ nonlinear frequency shift enters resonance!
here:

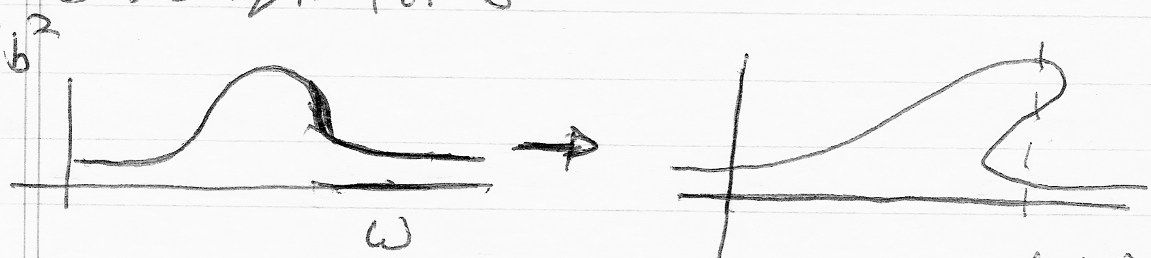
$$\omega-\omega_0 \rightarrow \omega-\omega_0 - K b^2$$

{amplitude-dependent shift}

⇒

$$b^2 = \left(\frac{f^2}{4m^2\omega_0^2}\right) 1 \Big/ \left[(\omega-\omega_0-Kb^2)^2+\alpha^2\right]$$

→ cubic eqn. for b^2

⇒ 

(symmetric) (asymmetric)

S-curve, bifurcations, mode jumping, etc.

6.) Nonlinear Oscillator with ± Dissipation

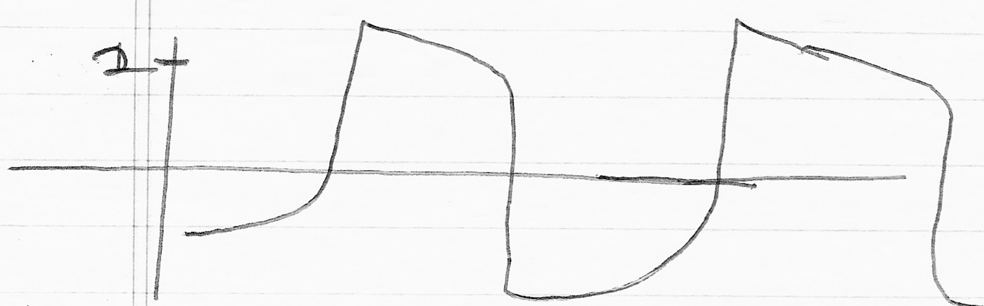
e.g. Van-der-Pol oscillator/Eqn.:

$$\ddot{x} + \alpha(x^2-1)\dot{x} + \omega_0^2 x = 0$$

key feature: nonlinear variable sign > dissipation (friction)

$$\Rightarrow \begin{cases} x^2 < 1 \rightarrow \alpha(x^2-1) < 0 \Rightarrow \begin{cases}\text{negative dissipation} \\ \text{instability}\end{cases} \\ \\ x^2 > 1 \rightarrow \alpha(x^2-1) > 0 \Rightarrow \begin{cases}\text{positive dissipation} \\ \rightarrow \text{stability / saturation}\end{cases} \end{cases}$$

$\Rightarrow$ bursty behavior, characteristic of limit cycle (at $x^2 = 1$)



Note:
In contrast to previous examples, system is fundamentally dissipative – nonlinearity in friction.

System/Eqn.	Nonlinearity	Externals
Mathieu (Parametric)	None	$\omega_0^2 = \omega_0^2(t)$ fast
Duffing (Freq. Shift)	Potential	None
Forced Duffing (Bi-stability / Bifurcation)	Potential	Forcing
Van-der-Pol (Limit Cycle)	Friction	None